

# Quantum information cannot be split into complementary parts

D.L. Zhou <sup>a,b,\*</sup>, B. Zeng <sup>c</sup>, L. You <sup>a,b</sup>

<sup>a</sup>*School of Physics, Georgia Institute of Technology, Atlanta, GA 30332, USA*

<sup>b</sup>*Institute of Theoretical Physics, The Chinese Academy of Sciences, Beijing, 100080, China*

<sup>c</sup>*Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

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## Abstract

We prove a new impossibility for quantum information (the no-splitting theorem): an unknown quantum bit (qubit) cannot be split into two complementary qubits. This impossibility, together with the no-cloning theorem, demonstrates that an unknown qubit state is a single entity, which cannot be cloned or split. This sheds new light on quantum computation and quantum information.

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## 1 Introduction

As a century old theory, quantum mechanics has provided the most effective description of the physical world. Recently, new discoveries were found for its applications to information and computation science [1], *e.g.*, the efficient prime factorization of larger numbers [2] and the perfectly secure quantum cryptography [3]. These, and related developments, have highlighted a general theme that quantum mechanics often makes impossible tasks in the classical world possible. Conversely, some possible operations in the classical world become impossible in the quantum world [4]. For example, an unknown quantum

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\* Corresponding author.

*Email addresses:* dz34@mail.gatech.edu (D.L. Zhou), zengbei@MIT.EDU (B. Zeng), ly14@mail.gatech.edu (L. You).

state cannot be perfectly cloned [5,6], while copies of an unknown quantum state cannot be deleted except for being swapped into the subspace of an ancilla [7].

The principle of linear superposition of states is an important feature of quantum mechanics. A significant consequence is that an unknown quantum state cannot be perfectly cloned, which has been known for quite some time [5,6]. This impossibility can also be understood from the causality requirement that no signal can be transmitted faster than the speed of light, even with the aid of nonlocal quantum resource such as entanglement. With the rapid development of quantum information science in recent years, we have come to realize the essential role of this simple, yet profound, limitation in quantum information processing, especially in quantum cryptography [3]. Intuitively, the no-cloning theorem implies there exists an essential difference between one copy and an ensemble of such copies of an unknown quantum state. One cannot obtain any information from only one copy of the quantum state without any prior knowledge of the state. Extensive research has focused on the no-cloning theorem related topics in quantum information science [8,9,10]. Recently, Pati discovered another important theorem of impossibilities for an unknown quantum state based on the principle of linear superposition: no linear transformations on two copies of an unknown quantum state can delete a copy except for being swapped into an ancilla state [7].

In this letter, we show that yet another theorem of impossibilities exists: quantum information of an unknown qubit cannot be split into two complementing qubits, *i.e.*, the information in one qubit is an inseparable entity. Our paper is organized as follows: in Sec. II we present our no-splitting problem in terms of a common scenario from quantum secret sharing. We show that if our discussion is restricted to only product pure final states, then the no-splitting statement is apparently valid. Following, in Sec. III, we consider the nontrivial case of the no-splitting problem, *i.e.*, for pure entangled final states. We then present a no-splitting theorem for a two-qubit case and argue that the no-splitting theorem also should be true in more general cases. Finally, we discuss several effects and applications of our no-splitting problem and point out possible future directions.

We note that Pati and Sanders have independently developed a similar idea – the no-partial erasure of quantum information – in a recent paper [11]. They claim that our non-splitting theorem becomes a straightforward corollary of their no-partial eraser theorem. This, however, is not the case. As demonstrated in their example of Eq.(8), if the final state is allowed to be a mixed state (for example due to entanglement with an ancilla), their no-partial eraser becomes invalid. On the contrary, the final pure state can contain entanglement between of the two (complementary) qubits for our theorem, thus our result must supersede their no-partial erasure theorem. In fact, as we show

in Sec. II, the no-partial erasure theorem is valid for product states, but not for the more general case of entangled states in Sec. III. We emphasize that the possible existence of entanglement between the two qubits is what makes our theorem on non-splitting of quantum information more important.

## 2 The No-splitting problem

We start by presenting our non-splitting idea in terms of a common scenario from quantum secret sharing: we assume that Alice and Bob want to store and share a secret, say, an unknown spatial direction of a qubit on the Bloch sphere, specified by its Euler angle  $(\theta, \phi)$ . If this secret is initially held by Alice, she can simply send the unknown value of  $\theta$  or  $\phi$  to Bob in the classical world, and this would accomplish one simple scheme of the secret sharing as they now each possess the complementary part of the secret  $\theta$  or  $\phi$ . However, this scheme as well as all other classically allowed more sophisticated schemes is impossible in the quantum world.

With the pseudo-spin representation on the Bloch sphere, the unknown qubit initially held by Alice can be denoted as

$$|v(\theta, \phi)\rangle_A = \cos \frac{\theta}{2} |0\rangle_A + \sin \frac{\theta}{2} e^{i\phi} |1\rangle_A. \quad (1)$$

In terms of this state, the no-cloning theorem says that there exists NO unitary transformation  $\mathcal{U}$  such that

$$\mathcal{U}|v(\theta, \phi)\rangle_A |w\rangle_B = |v(\theta, \phi)\rangle_A |v(\theta, \phi)\rangle_B, \quad (2)$$

where  $|w\rangle_B$  denotes an arbitrary given state of the ancilla qubit  $B$ . The no-deleting theorem of Pati states that there exists NO unitary transformation  $\mathcal{U}$  either to achieve the following

$$\mathcal{U}|v(\theta, \phi)\rangle_A |v(\theta, \phi)\rangle_B |w\rangle_C = |v(\theta, \phi)\rangle_A |x\rangle_B |y\rangle_C, \quad (3)$$

where for clarity we have assumed two copies of the unknown state. And,  $|x\rangle_B$  and  $|y\rangle_C$  are any known states.

A restricted form of the no-splitting theorem, *the two real parameters  $(\theta, \phi)$  contains in one qubit can not be split into two complementary qubits in a product state*, can be mathematically stated as follows. There does not exist any unitary transformation  $\mathcal{U}$  such that

$$|\Psi\rangle_{AB} := \mathcal{U}|v(\theta, \phi)\rangle_A |w\rangle_B = |x(\theta)\rangle_A |y(\phi)\rangle_B. \quad (4)$$

When we use the linearity of  $\mathcal{U}$  (from quantum mechanics), the plausible forms for states on the right hand side of Eq. (4) are

$$|x(\theta)\rangle_A = \cos \frac{\theta}{2} |x_1\rangle_A + \sin \frac{\theta}{2} |x_2\rangle_A, \quad (5)$$

$$|y(\phi)\rangle_B = |y_1\rangle_B + e^{i\phi} |y_2\rangle_B, \quad (6)$$

with un-normalized states  $|x_1\rangle_A$ ,  $|x_2\rangle_A$ ,  $|y_1\rangle_B$ , and  $|y_2\rangle_B$ , all independent of  $\theta$  and  $\phi$ . It is an easy exercise to conclude this kind of linear transformation cannot exist in quantum mechanics by comparing the LHS with the RHS of Eq. (4).

The above version of no-splitting theorem for product pure final states is valid also for more general cases with higher dimensions and more parameters. This restricted version can indeed be derived from the no-partial erasure theorem (Theorem 4) in Ref. [11], but the converse is not true (Corollary 5 in Ref. [11]). We will show in the following section that the no-partial erasure theorem is invalid for the more general case of entangled pure final states. In contrast, our no-splitting theorem remains valid for both cases.

### 3 No-splitting theorem

The above restricted version of the theorem is limited to separable pure states in the RHS of Eq. (4). More generally,  $|\Psi\rangle_{AB}$  can take the form of an entangled pure state. For example, when the unitary transformation  $\mathcal{U}$  corresponds to a control-NOT gate with qubit  $A$  as the control qubit and  $|w\rangle_B = |0\rangle_B$ , we obtain

$$\begin{aligned} |\Psi\rangle_{AB} = & \frac{1}{2} \left( \cos \frac{\theta}{2} |0\rangle_A + \sin \frac{\theta}{2} |1\rangle_A \right) (|0\rangle_B + e^{i\phi} |1\rangle_B) \\ & + \frac{1}{2} \left( \cos \frac{\theta}{2} |0\rangle_A - \sin \frac{\theta}{2} |1\rangle_A \right) (|0\rangle_B - e^{i\phi} |1\rangle_B), \end{aligned} \quad (7)$$

which consists of coherent superpositions where each contains a split state of  $\theta$  and  $\phi$ . Does this example point to a failure of our non-splitting idea when  $|\Psi\rangle_{AB}$  is an entangled state? No. In fact, in this case we only need to examine the reduced density matrix of qubit  $A$  and  $B$ , respectively. For the state (7), the reduced density matrix for qubit  $A$  (or  $B$ ) is

$$\rho_{A(B)} = \cos^2 \frac{\theta}{2} |0\rangle_{A(B)} \langle 0| + \sin^2 \frac{\theta}{2} |1\rangle_{A(B)} \langle 1|, \quad (8)$$

both independent of  $\phi$ . Thus, the above example does not provide a counterexample to our non-splitting idea.

It is also straightforward to show that the no-partial erasure theorem of Pati and Sanders [11] is no longer valid in this case, since for the state (7), simply discarding one qubit will result in a mixed state with parameter  $\theta$ . This observation is trivial because a simple measurement in the computational basis will erase the information of  $\phi$ . On the other hand, as shown by the above observation, our no-splitting theorem remains valid. We formulated our no-splitting idea into the following theorem, which constitutes the central result of this letter.

**Theorem 1** *There exists no two-qubit unitary transformation  $\mathcal{U}$  capable of splitting an unknown qubit. In mathematical terms, the transformed state is*

$$|\Psi\rangle_{AB} := \mathcal{U}|v(\theta, \phi)\rangle_A|w\rangle_B, \quad (9)$$

where  $|v(\theta, \phi)\rangle_A$  is defined in Eq. (1), and  $|w\rangle_B$  is an arbitrarily given pure state of qubit  $B$ . This theorem then states that

$$\text{tr}_B(|\Psi\rangle_{ABAB}\langle\Psi|) = \rho_A(\theta) \quad (10)$$

and

$$\text{tr}_A(|\Psi\rangle_{ABAB}\langle\Psi|) = \rho_B(\phi) \quad (11)$$

cannot be satisfied simultaneously.

We now prove this general result.

**Proof:** Inserting Eq. (1) into Eq. (9), we obtain

$$|\Psi\rangle_{AB} = \cos \frac{\theta}{2} \mathcal{U}|0\rangle_A|w\rangle_B + \sin \frac{\theta}{2} e^{i\phi} \mathcal{U}|1\rangle_A|w\rangle_B. \quad (12)$$

Applying the Schmidt decomposition of a two-qubit pure state, we immediately find

$$\mathcal{U}|1\rangle_A|w\rangle_B = r_0|\tilde{0}\tilde{0}\rangle_{AB} + r_1|\tilde{1}\tilde{1}\rangle_{AB}, \quad (13)$$

where  $|\tilde{0}\rangle_{A(B)}$  and  $|\tilde{1}\rangle_{A(B)}$  are the corresponding orthogonal basis states of the Schmidt decomposition for qubits  $A$  and  $(B)$ , and  $r_0$  and  $r_1$  are real parameters which satisfy the normalization condition

$$r_0^2 + r_1^2 = 1. \quad (14)$$

Because the state  $\mathcal{U}|0\rangle_A|w\rangle_B$  is orthogonal to state  $\mathcal{U}|1\rangle_A|w\rangle_B$ , we deduce that

$$\mathcal{U}|0\rangle_A|w\rangle_B = \alpha r_1 |\tilde{0}\tilde{0}\rangle_{AB} - \alpha r_0 |\tilde{1}\tilde{1}\rangle_{AB} + c |\tilde{0}\tilde{1}\rangle_{AB} + d |\tilde{1}\tilde{0}\rangle_{AB}, \quad (15)$$

where  $\alpha$ ,  $c$ , and  $d$  are generally complex. They satisfy the normalization condition

$$|\alpha|^2 + |c|^2 + |d|^2 = 1. \quad (16)$$

The conditions of Eqs. (10) and (11) are summarized in the following equivalent set of equations:

$$d^* r_0 = 0, \quad (17)$$

$$c r_1 = 0, \quad (18)$$

$$c^* r_0 = 0, \quad (19)$$

$$d r_1 = 0, \quad (20)$$

$$\alpha r_0 r_1 = 0, \quad (21)$$

$$|\alpha|^2 r_1^2 + |d|^2 - r_0^2 = 0, \quad (22)$$

$$c^* \alpha r_1 - d \alpha^* r_0 = 0. \quad (23)$$

Suppose  $r_0 \neq 0$ , then  $c = d = \alpha r_1 = 0$ , but  $r_0^2 = |\alpha|^2 r_1^2 + |d|^2 = 0$ ; therefore,  $r_0 = 0$ , which is contradictory. Now assume  $r_0 = 0$ , which leads to  $r_1 \neq 0$  and  $c = d = 0$ , then  $|\alpha|^2 = (r_0^2 - |d|^2)/r_1^2 = 0$ , thus  $|\alpha|^2 + |c|^2 + |d|^2 = 0$ . Again this is contradictory. Thus, there is no self-consistent solution to Eqs. (10) and (11), *i.e.*, we have completed the proof of our theorem.

When  $|\Psi\rangle_{AB}$  is a product pure state, Eqs. (10) and (11) reduces to Eq. (4). Theorem 1 further indicates that the information of the amplitude ( $\theta$ ) and the phase ( $\phi$ ) cannot be split into two qubits by any two-qubit unitary transformation, even for more general (entangled) pure final two qubit states.

We speculate that the no-splitting theorem is valid for more general cases of higher dimensional Hilbert spaces with more parameters. This is based on the observation that the number of constraining equations grows faster than the number of parameters; hence, in general no solution could be expected just as we show above for the case of two qubits.

## 4 Applications and future directions

It has been debated that some tasks of quantum information processing can only be implemented in real Hilbert space or restricted to equatorial states

(states with the same amplitude on all the computational basis but different phases). However, the tasks never would work in the complete complex Hilbert space, for example, Pati's remote state preparation protocol [12] and its higher dimensional generalizations [13], the (2, 2) quantum secret sharing protocol with pure states [14], and Yao's self-testing quantum apparatus [15]. Our theorem, therefore, provides a stronger evidence that all such tasks can never be implemented in the whole complex Hilbert space, even including the potential effort of transferring complex states into real or equatorial ones. Furthermore, Grover's algorithm [16] only calls for rotations of real angles, and Shor's algorithm [17] requires discrete Fourier transform which only needs transformation between equatorial states. Our theorem thus implies that in some cases, the restricted quantum information and computation schemes in real or equatorial space may have the same power [18], or even more power, than schemes in the whole complex Hilbert space.

Interestingly, despite such strong restrictions from the restricted version of our no-splitting theorem or the no-partial erasure theorem [11] that there exists even no probabilistic approach for splitting or partially erasing an unknown state, the converse procedure, *i.e.*, to combine two states

$$\cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |1\rangle, \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi} |1\rangle) \quad (24)$$

into one can be easily accomplished. As a simple example, we give the following protocol starting from

$$\left( \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} |1\rangle \right) \otimes \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi} |1\rangle), \quad (25)$$

executing a parity detection measurement ( $ZZ$ ), followed by an XOR gate, then discarding the ancillary qubit, we will reach either

$$\cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\varphi} |1\rangle, \quad (26)$$

or

$$\cos \frac{\theta}{2} e^{i\varphi} |0\rangle + \sin \frac{\theta}{2} |1\rangle, \quad (27)$$

both with the probability of 1/2. We believe this interesting observation will shed light on future investigations of the “quantum nature” of quantum information.

In summary, we have shown that the unknown information of one copy of a

qubit cannot be split into two complementary qubits, whether the final pure state of the two qubits is separable or entangled. Our result demonstrates the inseparable property for quantum information in terms of an unknown single qubit and is schematically illustrated in Fig. 1. Together with the no-cloning theorem, the no-splitting theorem shows that one qubit is an entity that corresponds to the basic unit in quantum computation and quantum information.

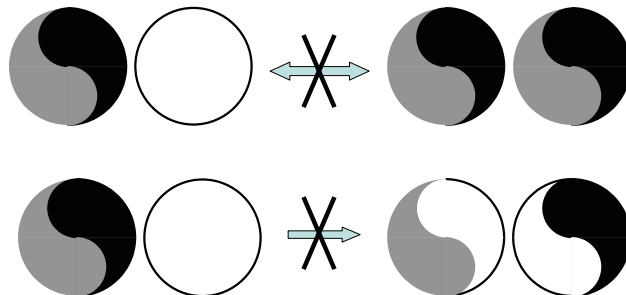


Fig. 1. Schematic illustration of our result for no-splitting is in the second row, as compared to the no-cloning and its inverse no-deleting theorems in the first row. The unknown initial qubit is represented by the ying-yang circle together with the known ancilla qubit represented by the empty circle on the left.

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## References

- [†] Current address: Department of Physics, Massachusetts Institute of Technology, MA 02139, USA.
- [1] M.A. Nielsen and I.L. Chuang, *Quantum computation and quantum information*, (Cambridge University Press, 2000).
- [2] P.W. Shor, in *Proc. 37<sup>th</sup> Annual Symposium on Fundamentals of Computer Science*, pp. 56-65, (IEEE press, Los Alamitos, CA 1996).
- [3] C.H. Bennett and G. Brassard, in *Proc. of IEEE International Conference on Computers, Systems and Signal Processing*, pp. 175-179, (IEEE, New York, 1984).
- [4] A.K. Pati, Phys. Rev. A **66**, 062319 (2002).
- [5] W.K. Wootters and W.H. Zurek, Nature **99**, 802 (1982).
- [6] D. Dieks, Phys. Lett. A **92**, 271 (1982).
- [7] A.K. Pati and S.L. Braunstein, Nature **404**, 164 (2000).



- [8] H.P. Yuen, Phys. Lett. A **113**, 405 (1986).
- [9] H. Barnum, C.M. Caves, C.A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. **76**, 2818 (1996).
- [10] L.M. Duan and G.C. Guo, Phys. Rev. Lett. **80**, 4999 (1998).
- [11] A. K. Pati, and B. C. Sanders, quant-ph/0503138.
- [12] A.K. Pati, Phys. Rev. A **63**, 014302 (2001).
- [13] B. Zeng and P. Zhang, Phys. Rev. A **65**, 022316 (2001).
- [14] R. Cleve, D. Gottesman, and H.K. Lo, Phys. Rev. Lett. **83**, 648 (1999).
- [15] D. Mayers and A. Yao, (quant-ph/0307205).
- [16] L.K. Grover, Phys. Rev. Lett. **78**, 325 (1997).
- [17] P. W. Shor, in *Proc. 35<sup>th</sup> Annual Symposium on the Foundations of Computer Science*, pp. 124-134, (IEEE Computer Society Press, New York, 1994).
- [18] E. Bernstein and U. Vazirani, SIAM J. Comput. **26**, 1411 (1997).